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VIBRATION OF GEOMETRICALLY IMPERFECT BEAM AND SHELL STRUCTURES

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Abstract—The purpose of this paper is to present a general theory for analysis of the effect of initial geometrical imperfections on vibration frequencies of undamped, conservatively loaded, linear elastic beam and shell structures. The theory will be restricted to single mode vibrations with imperfections in the same shape as the vibration modes. The mathematical tool is a perturbational procedure developed with the aid of the principle of virtual work. The approach is illustrated by applications to beams, plates and axisymmetric shell structures. The examples show that the vibration frequency of these structures may be significantly raised or lowered due to imperfections.

NOTATION

a, b	length and width of plate (Fig. 4)
A	area of beam cross-section
a_1	coefficient for imperfection sensitivity of nonsymmetric structures [eqns (29), (30)]
b_1, b_2, b_3, b_4	coefficients for imperfection sensitivity of symmetric structures [eqns (35), (36)]
c	coefficient for vibration mode [eqns (29), (35)]
d	length of beam
D	flexural stiffness of plate [eqn (63)]
E	Young's modulus
h	thickness of plate or conical shell
m, n	bending moment and axial force in beam (functions of location)
m_{xx}, m_{xy}, m_{yy}	bending moments in plate (functions of location)
n_{xx}, n_{xy}, n_{yy}	in-plane stresses in plate (functions of location)
n_{ap}	applied stress at plate
n_c	circumferential wave number for conical shell vibration mode
P	axial load at conical shell
P_r	equivalent force in beam or plate
r	smaller end radius of conical shell (Fig. 6)
r_g	radius of gyration for beam
R	larger end radius of conical shell (Fig. 6)
t	time coordinate
$\mathbf{u}$	generalized displacements (vector function of location)
u, v, w	displacements (functions of location)
v_d	forced displacement of beam end (Fig. 2)
x, y, z	Cartesian coordinate system
α	nonlinearity coefficient [eqn (79)]
β	semi vertex angle for conical shell (Fig. 6)
$\boldsymbol{\varepsilon}$	generalized strains (vector function of location)
ε, κ	axial strain and curvature for beam (functions of location)
$\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$	in-plane strains for plate (functions of location)
$\kappa_{xx}, \kappa_{xy}, \kappa_{yy}$	bending curvatures for plate (functions of location)
λ_s, λ_c	load factors [eqn (79)]
ρ, ν	mass density and Poisson's ratio
$\bar{\xi}, \xi_0, \xi_p, \xi_r$	mode amplitude parameters
$\boldsymbol{\sigma}$	generalized stresses (vector function of location)
ω, ω_p	vibration frequency of imperfect and perfect structure
ω_r	non-dimensional vibration frequency [eqns (56), (79)]

Operators

H	stress–strain operator [eqns (6), (7)]
L_1, L_2, L_{11}	strain–deformation operators [eqns (3)–(5)]
M	mass-deformation operator [eqn (2)]
δ	linear variational operator
Φ	Airy's stress function [eqn (64)]
∇^2, ∇^4	differential operators [eqn (64)]
$(\ )_{,a}$	differentiation with respect to a
$(\ddot{\ })$	double differentiation with respect to time

Indices

$(\bar{\ })$	*response of the perfect structure*
$(\tilde{\ })$	initial imperfection
$(\ )_{is}$, $i = 1, 2, \ldots$	perturbation terms for the steady state
$(\ )_{iv}$, $i = 1, 2, \ldots$	perturbation terms for the vibrating state
$(\ )_1$	common vibration and mode shape [eqn (19)]
$(\ )_\alpha, (\ )_\beta$	two different states.

1. INTRODUCTION

In the present paper a theory for small vibrations of conservatively loaded imperfect structures is developed using the methodology which has been applied to initial postbuckling analysis by Budiansky (1966) and Fitch (1968) among others.

The initial postbuckling theory developed by Koiter (1967) and rederived by Budiansky (1966) and Fitch (1968) using a virtual work approach is concerned with the effect of small initial imperfections on the load-carrying capacity of structures. This postbuckling theory has been used to investigate buckling, but is also capable of treating vibrations of structures under static loads. Rehfield (1973) has treated large amplitude vibrations of elastic structures using a theory, which is analogous to the postbuckling theory, using Hamilton's principle. However, static external loading and initial imperfections are not taken into account. The influence of imperfections on the nonlinear vibrations of beams and plates has recently been investigated theoretically by Elishakoff *et al.* (1985), Hui and Leissa (1983), Hui (1984a) and Ilanko and Dickinson (1987a), respectively. Vibrations of imperfect plates have also been investigated experimentally by Ilanko and Dickinson (1987b). Furthermore, Liu and Arbocz (1986a,b) have carried out a comprehensive study of the influence of initial geometric imperfections on the non-linear vibration behaviour of undamped and damped circular cylindrical shells, and Elishakoff *et al.* (1987) and Hui (1984b) have been investigating vibrations of imperfect cylindrical panels.

The present analysis deals with the influence of an initial geometrical imperfection on the vibration frequency of a structure, at a given conservative load. The imperfection is assumed to be of the same shape as the vibration mode, as this is expected to represent the most interesting case. The amplitude of the imperfection is included in the theory such that also the imperfection sensitivity of the unloaded structure can be investigated. The notation used in the present analysis is similar to the notation used by Budiansky (1966), Fitch (1968) and Rehfield (1973). Detailed information on the interpretation of the notation in case of axisymmetric shells can be found in Fitch (1968) and in Pedersen and Jensen (1976).

Two analytical examples, a beam and a rectangular plate, are used to illustrate the theory. The results from the present approach are in accordance with previously published results by Elishakoff *et al.* (1985) and Hui and Leissa (1983). A numerical example concerning a truncated conical shell is used to demonstrate the application of computer-based imperfection sensitivity analysis.

2. GOVERNING EQUATIONS

In order to account for inertia forces, the equation of equilibrium may be formulated as:

$$M(\ddot{\mathbf{u}}) \cdot \delta\mathbf{u} + \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon} = \mathbf{q} \cdot \delta\mathbf{u}. \tag{1}$$

Here $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ and $\mathbf{q}$ denote generalized displacements, strains, stresses and static loads, respectively. The symbols can be thought of as denoting vector functions. The internal virtual work of the stress $\boldsymbol{\sigma}$ through a strain variation $\delta\boldsymbol{\varepsilon}$ integrated over the entire structure is denoted by $\boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon}$. In the same way the external virtual work of the load $\mathbf{q}$, and the virtual work of the d'Alembert forces $-M(\ddot{\mathbf{u}})$, through a kinematic admissible displacement variation $\delta\mathbf{u}$ is denoted by $-M(\ddot{\mathbf{u}}) \cdot \delta\mathbf{u}$ and $\mathbf{q} \cdot \delta\mathbf{u}$, respectively. The overdot denotes differentiation with respect to time and the generalized mass operator M is assumed linear in $\ddot{\mathbf{u}}$. Equation (1) will here be employed for a "frozen-in-time picture" of the structure at the instant when the amplitude of the vibration attains its maximum. The mass operator is assumed to have the property that

$$M(\mathbf{u}_a)\cdot\mathbf{u}_b = M(\mathbf{u}_b)\cdot\mathbf{u}_a \tag{2}$$

for any kinematic admissible value of $\mathbf{u}_b$ and $\mathbf{u}_a$.

For a perfect structure the strain may be written as

$$\boldsymbol{\varepsilon} = L_1(\mathbf{u})+\tfrac{1}{2}L_2(\mathbf{u}), \tag{3}$$

where L_1 and L_2 are linear and quadratic functionals, respectively.

For a structure with an initial geometric imperfection $\hat{\mathbf{u}}$ the strain becomes

$$\boldsymbol{\varepsilon} = L_1(\hat{\mathbf{u}}+\mathbf{u})+\tfrac{1}{2}L_2(\hat{\mathbf{u}}+\mathbf{u})-L_1(\hat{\mathbf{u}})-\tfrac{1}{2}L_2(\hat{\mathbf{u}}) = L_1(\mathbf{u})+\tfrac{1}{2}L_2(\mathbf{u})+L_{11}(\hat{\mathbf{u}},\mathbf{u}), \tag{4}$$

where the bilinear functional L_{11} is defined by

$$L_2(\mathbf{u}_a+\mathbf{u}_b) = L_2(\mathbf{u}_a)+2L_{11}(\mathbf{u}_a,\mathbf{u}_b)+L_2(\mathbf{u}_b). \tag{5}$$

The linear stress–strain relation

$$\boldsymbol{\sigma} = H(\boldsymbol{\varepsilon}), \tag{6}$$

is assumed to have the property that

$$H(\boldsymbol{\varepsilon}_a)\cdot\boldsymbol{\varepsilon}_b = H(\boldsymbol{\varepsilon}_b)\cdot\boldsymbol{\varepsilon}_a. \tag{7}$$

2.1. *Vibration analysis*

Let us now study the effects of initial imperfections and a conservative external loading on the vibration behaviour of slender structures. We will use $\bar{\mathbf{u}}$ to describe the deformations of a perfect structure at a given load. It is well known that imperfections can initiate deformation in a buckling mode, even if the load is well below the critical load for bifurcation buckling. Provided that the structure has a small geometrical imperfection with the mode shape $\mathbf{u}_{1s}$ and the amplitude $\hat{\xi}$, the deformations will consist of a small static part with the mode shape $\mathbf{u}_{1s}$ and the amplitude ξ_s in addition to $\bar{\mathbf{u}}$. For larger imperfections the deflection in the buckling mode is known to be nonlinear. For this reason a perturbational expansion of the buckling deflections will be employed. Under free vibrations the deformations of a perfect structure will furthermore include a part that varies harmonically in time with the mode shape $\mathbf{u}_{1v}$, the amplitude ξ_v and the circular frequency ω. For the imperfect structure the vibration mode will depend on the imperfection. In the following we will restrict the analysis to small vibration amplitudes. Higher order terms of ξ_v and terms, which vary with frequencies that are multiples of ω, are therefore neglected.

The initial geometric imperfections is taken as a combination of components $\mathbf{u}_{1s}$, $\mathbf{u}_{2s}$, $\mathbf{u}_{3s}$, ... of the nonlinear deformations in the steady state:

$$\hat{\mathbf{u}} = \hat{\xi}\mathbf{u}_{1s}+\hat{\xi}^2\mathbf{u}_{2s}+\hat{\xi}^3\mathbf{u}_{3s}+\cdots. \tag{8}$$

Then the perturbational expansion of the deformations can be written in the form

$$\mathbf{u} = \bar{\mathbf{u}}+(\xi_t-\hat{\xi})\mathbf{u}_{1s}+(\xi_t^2-\hat{\xi}^2)\mathbf{u}_{2s}+(\xi_t^3-\hat{\xi}^3)\mathbf{u}_{3s}+\cdots+\xi_v\cos\omega t[\mathbf{u}_{1v}+\xi_t\mathbf{u}_{2v}+\xi_t^2\mathbf{u}_{3v}+\cdots] \tag{9}$$

where the nondimensional mode amplitude parameter ξ_t is the sum of the amplitude ξ_s of the deformation in the mode $\mathbf{u}_{1s}$ and the imperfection amplitude parameter $\hat{\xi}$,

$$\xi_t = \hat{\xi}+\xi_s. \tag{10}$$

The deformations are measured relative to the imperfect structure. The perturbational expansion of the deformations are chosen so that the sum of the deformations and the imperfection is an exponential expansion in the mode amplitude parameter ξ_s. We shall later see that this expansion is consistent, since using eqns (3) and (6) the perturbational expansions of the strains and the stresses are found to have the same form as (9).

The quantities $\mathbf{u}_{2s}$, $\mathbf{u}_{2r}$, $\mathbf{u}_{3s}$ and $\mathbf{u}_{3r}$ can be interpreted as corrections to $\mathbf{u}_{1s}$ and $\mathbf{u}_{1r}$ for finite values of ξ_s and $\bar{\xi}$ in order to comply with boundary conditions etc. Thereby, the effect of the amplitude $\bar{\xi}$ of the imperfections on the total deflection $\mathbf{u}$ is described only through the static buckling amplitude parameter ξ_s and as seen later, also the frequency ω. In order to make the expansions unique, the displacement components $\mathbf{u}_{2s}$, $\mathbf{u}_{3s}$, ... are orthogonalized with respect to $\mathbf{u}_{1s}$ and the displacement increments $\mathbf{u}_{2r}$, $\mathbf{u}_{3r}$, ... are orthogonalized with respect to $\mathbf{u}_{1r}$

$$M(\mathbf{u}_{is})\cdot\mathbf{u}_{1s} = 0 \quad \text{and} \quad M(\mathbf{u}_{ir})\cdot\mathbf{u}_{1r} = 0 \quad \text{for} \quad i = 2, 3, \ldots \tag{11}$$

without loss of generality.

Using eqn (4) the perturbational expansions of strains and stresses of the structure become

$$\begin{aligned}
\boldsymbol{\varepsilon} &= \bar{\boldsymbol{\varepsilon}} + (\xi_s - \bar{\xi})\boldsymbol{\varepsilon}_{1s} + (\xi_s^2 - \bar{\xi}^2)\boldsymbol{\varepsilon}_{2s} + (\xi_s^3 - \bar{\xi}^3)\boldsymbol{\varepsilon}_{3s} + \cdots + \xi_r \cos\omega t[\boldsymbol{\varepsilon}_{1r} + \xi_s\boldsymbol{\varepsilon}_{2r} + \xi_s^2\boldsymbol{\varepsilon}_{3r} + \cdots] \\
\boldsymbol{\sigma} &= \bar{\boldsymbol{\sigma}} + (\xi_s - \bar{\xi})\boldsymbol{\sigma}_{1s} + (\xi_s^2 - \bar{\xi}^2)\boldsymbol{\sigma}_{2s} + (\xi_s^3 - \bar{\xi}^3)\boldsymbol{\sigma}_{3s} + \cdots + \xi_r \cos\omega t[\boldsymbol{\sigma}_{1r} + \xi_s\boldsymbol{\sigma}_{2r} + \xi_s^2\boldsymbol{\sigma}_{3r} + \cdots]
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\bar{\boldsymbol{\varepsilon}} &= L_1(\bar{\mathbf{u}}) + \tfrac{1}{2}L_2(\bar{\mathbf{u}}) \\
\boldsymbol{\varepsilon}_{1s} &= L_1(\mathbf{u}_{1s}) + L_{11}(\bar{\mathbf{u}}, \mathbf{u}_{1s}) \\
\boldsymbol{\varepsilon}_{2s} &= L_1(\mathbf{u}_{2s}) + L_{11}(\bar{\mathbf{u}}, \mathbf{u}_{2s}) + \tfrac{1}{2}L_2(\mathbf{u}_{1s}) \\
\boldsymbol{\varepsilon}_{3s} &= L_1(\mathbf{u}_{3s}) + L_{11}(\mathbf{u}_{1s}, \mathbf{u}_{2s}) + L_{11}(\bar{\mathbf{u}}, \mathbf{u}_{3s}) \\
\boldsymbol{\varepsilon}_{1r} &= L_1(\mathbf{u}_{1r}) + L_{11}(\bar{\mathbf{u}}, \mathbf{u}_{1r}) \\
\boldsymbol{\varepsilon}_{2r} &= L_1(\mathbf{u}_{2r}) + L_{11}(\bar{\mathbf{u}}, \mathbf{u}_{2r}) + L_{11}(\mathbf{u}_{1r}, \mathbf{u}_{1s}) \\
\boldsymbol{\varepsilon}_{3r} &= L_1(\mathbf{u}_{3r}) + L_{11}(\mathbf{u}_{1r}, \mathbf{u}_{2s}) + L_{11}(\mathbf{u}_{2r}, \mathbf{u}_{1s}) + L_{11}(\bar{\mathbf{u}}, \mathbf{u}_{3r}).
\end{aligned} \tag{13}$$

From eqn (6) the stresses of the structure are found as

$$\bar{\boldsymbol{\sigma}} = H(\bar{\boldsymbol{\varepsilon}}); \quad \boldsymbol{\sigma}_{is} = H(\boldsymbol{\varepsilon}_{is}) \quad \text{and} \quad \boldsymbol{\sigma}_{ir} = H(\boldsymbol{\varepsilon}_{ir}) \quad i = 1, 2, 3, \ldots. \tag{14}$$

It is from these equations that we can see that the initial geometric imperfection (8) together with the perturbational expansion of the deformations (9) fits with the perturbational expansion of the strains and the stresses (12).

Using eqns (4) and (9), the strain variation $\delta\boldsymbol{\varepsilon}$ associated with a small displacement increment $\delta\mathbf{u}$ of $\mathbf{u}$ becomes

$$\begin{aligned}
\delta\boldsymbol{\varepsilon} &= L_1(\delta\mathbf{u}) + L_{11}(\bar{\mathbf{u}}, \delta\mathbf{u}) + L_{11}(\dot{\mathbf{u}}, \delta\mathbf{u}) + (\xi_s - \bar{\xi})L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) \\
&\quad + (\xi_s^2 - \bar{\xi}^2)L_{11}(\mathbf{u}_{2s}, \delta\mathbf{u}) + (\xi_s^3 - \bar{\xi}^3)L_{11}(\mathbf{u}_{3s}, \delta\mathbf{u}) + \cdots \\
&\quad + \xi_r \cos\omega t[L_{11}(\mathbf{u}_{1r}, \delta\mathbf{u}) + \xi_s L_{11}(\mathbf{u}_{2r}, \delta\mathbf{u}) + \xi_s^2 L_{11}(\mathbf{u}_{3r}, \delta\mathbf{u}) + \cdots] \\
&= \delta\bar{\boldsymbol{\varepsilon}} + \xi_s L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) + \xi_s^2 L_{11}(\mathbf{u}_{2s}, \delta\mathbf{u}) + \xi_s^3 L_{11}(\mathbf{u}_{3s}, \delta\mathbf{u}) + \cdots \\
&\quad + \xi_r \cos\omega t[L_{11}(\mathbf{u}_{1r}, \delta\mathbf{u}) + \xi_s L_{11}(\mathbf{u}_{2r}, \delta\mathbf{u}) + \xi_s^2 L_{11}(\mathbf{u}_{3r}, \delta\mathbf{u}) + \cdots]
\end{aligned} \tag{15}$$

where

$$\delta\bar{\boldsymbol{\varepsilon}} = L_1(\delta\mathbf{u}) + L_{11}(\bar{\mathbf{u}}, \delta\mathbf{u}). \tag{16}$$

Inserting eqns (12) and (15) in the equation of equilibrium (1) the following two equations are derived. For the steady state part we get

$$\begin{aligned}\bar{\boldsymbol{\sigma}}\cdot\delta\bar{\boldsymbol{\varepsilon}} + \xi_t\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) + (\xi_t - \bar{\xi})\boldsymbol{\sigma}_{1s}\cdot\delta\bar{\boldsymbol{\varepsilon}} + \xi_t^2\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{2s}, \delta\mathbf{u}) + (\xi_t - \bar{\xi})\xi_t\boldsymbol{\sigma}_{1s}\cdot L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) \\ + (\xi_t^2 - \bar{\xi}^2)\boldsymbol{\sigma}_{2s}\cdot\delta\bar{\boldsymbol{\varepsilon}} + \xi_t^3\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{3s}, \delta\mathbf{u}) + (\xi_t - \bar{\xi})\xi_t^2\boldsymbol{\sigma}_{1s}\cdot L_{11}(\mathbf{u}_{2s}, \delta\mathbf{u}) \\ + (\xi_t^2 - \bar{\xi}^2)\xi_t\boldsymbol{\sigma}_{2s}\cdot L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) + (\xi_t^3 - \bar{\xi}^3)\boldsymbol{\sigma}_{3s}\cdot\delta\bar{\boldsymbol{\varepsilon}} + \cdots = \mathbf{q}\cdot\delta\mathbf{u}\end{aligned} \tag{17}$$

and for the vibrating state part with the common multiplier $\xi_v \cos\omega t$ we get

$$\begin{aligned}&-\omega^2 M(\mathbf{u}_{1v})\cdot\delta\mathbf{u} + \bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{1v}, \delta\mathbf{u}) + \boldsymbol{\sigma}_{1v}\cdot\delta\bar{\boldsymbol{\varepsilon}} - \xi_t\omega^2 M(\mathbf{u}_{2v})\cdot\delta\mathbf{u} + \xi_t\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{2v}, \delta\mathbf{u}) \\ &+ (\xi_t - \bar{\xi})\boldsymbol{\sigma}_{1s}\cdot L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) + \xi_t\boldsymbol{\sigma}_{1v}\cdot L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) + \xi_t\boldsymbol{\sigma}_{2v}\cdot\delta\bar{\boldsymbol{\varepsilon}} - \xi_t^2\omega^2 M(\mathbf{u}_{3v})\cdot\delta\mathbf{u} \\ &+ \xi_t^2\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{3v}, \delta\mathbf{u}) + (\xi_t - \bar{\xi})\xi_t\boldsymbol{\sigma}_{1s}\cdot L_{11}(\mathbf{u}_{2v}, \delta\mathbf{u}) + \xi_t^2\boldsymbol{\sigma}_{1v}\cdot L_{11}(\mathbf{u}_{2s}, \delta\mathbf{u}) \\ &+ (\xi_t^2 - \bar{\xi}^2)\boldsymbol{\sigma}_{2s}\cdot L_{11}(\mathbf{u}_{1v}, \delta\mathbf{u}) + \xi_t^2\boldsymbol{\sigma}_{2v}\cdot L_{11}(\mathbf{u}_{1s}, \delta\mathbf{u}) + \xi_t^2\boldsymbol{\sigma}_{3v}\cdot\delta\bar{\boldsymbol{\varepsilon}} + \cdots = 0.\end{aligned} \tag{18}$$

It is expected that the sensitivity of the vibration frequency to an imperfection in the vibration mode represents the most important case. Therefore, we restrict the analysis to the following case:

$$\mathbf{u}_{1s} = \mathbf{u}_{1v} = \mathbf{u}_1 \tag{19}$$

where $\mathbf{u}_1$ is the common mode shape. It is noted that eqn (19) only restricts the first-order part $\mathbf{u}_{1s}$ of the imperfection form.

Equations (16)–(19) yield with $\delta\mathbf{u} = \mathbf{u}_1$ the following two equations in ω and ξ_t

$$\begin{aligned}\xi_t[\bar{\boldsymbol{\sigma}}\cdot L_2(\mathbf{u}_1) + \boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}_1] - \bar{\xi}\boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}_1 + \xi_t^2[\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{2s}, \mathbf{u}_1) + \boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1) + \boldsymbol{\sigma}_{2s}\cdot\boldsymbol{\varepsilon}_1] \\ - \xi_t\bar{\xi}\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1) - \bar{\xi}^2\boldsymbol{\sigma}_{2s}\cdot\boldsymbol{\varepsilon}_1 + \xi_t^3[\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{3s}, \mathbf{u}_1) + \boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2s}, \mathbf{u}_1) + \boldsymbol{\sigma}_{2s}\cdot L_2(\mathbf{u}_1) + \boldsymbol{\sigma}_{3s}\cdot\boldsymbol{\varepsilon}_1] \\ - \bar{\xi}\xi_t^2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2s}, \mathbf{u}_1) - \bar{\xi}^2\xi_t\boldsymbol{\sigma}_{2s}\cdot L_2(\mathbf{u}_1) - \bar{\xi}^3\boldsymbol{\sigma}_{3s}\cdot\boldsymbol{\varepsilon}_1 + \cdots = 0\end{aligned} \tag{20}$$

and

$$\begin{aligned}[-\omega^2 M(\mathbf{u}_1)\cdot\mathbf{u}_1 + \bar{\boldsymbol{\sigma}}\cdot L_2(\mathbf{u}_1) + \boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}_1] + \xi_t[\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{2v}, \mathbf{u}_1) + 2\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1) + \boldsymbol{\sigma}_{2v}\cdot\boldsymbol{\varepsilon}_1] \\ - \bar{\xi}\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1) + \xi_t^2[\bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_{3v}, \mathbf{u}_1) + \boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2v}, \mathbf{u}_1) + \boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2s}, \mathbf{u}_1) + \boldsymbol{\sigma}_{2s}\cdot L_2(\mathbf{u}_1) \\ + \boldsymbol{\sigma}_{2v}\cdot L_2(\mathbf{u}_1) + \boldsymbol{\sigma}_{3v}\cdot\boldsymbol{\varepsilon}_1] - \bar{\xi}\xi_t\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2v}, \mathbf{u}_1) - \bar{\xi}^2\boldsymbol{\sigma}_{2s}\cdot L_2(\mathbf{u}_1) + \cdots = 0\end{aligned} \tag{21}$$

where

$$\boldsymbol{\varepsilon}_1 = L_1(\mathbf{u}_1) + L_{11}(\bar{\mathbf{u}}, \mathbf{u}_1) \quad \text{and} \quad \boldsymbol{\sigma}_1 = H(\boldsymbol{\varepsilon}_1). \tag{22}$$

For the perfect structure in the prebuckling state, eqn (17) reduces to the governing equation

$$\bar{\boldsymbol{\sigma}}\cdot\delta\bar{\boldsymbol{\varepsilon}} = \mathbf{q}\cdot\delta\mathbf{u}. \tag{23}$$

A variational formulation of the vibrating state of the perfect structure in the prebuckling state is obtained from eqn (18)

$$-\omega_p^2 M(\mathbf{u}_1)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_1\cdot\delta\bar{\boldsymbol{\varepsilon}} + \bar{\boldsymbol{\sigma}}\cdot L_{11}(\mathbf{u}_1, \delta\mathbf{u}) = 0 \tag{24}$$

where ω_p denotes the vibration frequency of the perfect structure associated with the mode $\mathbf{u}_1$.

Keeping (11) in mind, and letting $\delta\mathbf{u}$ be equal to $\mathbf{u}_1$, $\mathbf{u}_{2s}$, $\mathbf{u}_{2v}$, $\mathbf{u}_{3s}$, $\mathbf{u}_{3v}$, . . . eqn (24) is used to simplify eqns (20) and (21) for the static and the vibrating states, which then can be written as:

static state,

$$\xi_s\omega_p^2 M(\mathbf{u}_1)\cdot\mathbf{u}_1-\bar{\xi}\boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}_1+\tfrac{3}{2}\xi_s^2\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1)-\xi_s\bar{\xi}\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1)-\bar{\xi}^2\boldsymbol{\sigma}_{2s}\cdot\boldsymbol{\varepsilon}_1$$
$$+\xi_s^3[2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2s},\mathbf{u}_1)+\boldsymbol{\sigma}_{2s}\cdot L_2(\mathbf{u}_1)]-\bar{\xi}\xi_s^2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2s},\mathbf{u}_1)-\bar{\xi}^2\xi_s\boldsymbol{\sigma}_{2s}\cdot L_2(\mathbf{u}_1)$$
$$-\bar{\xi}^3\boldsymbol{\sigma}_{3s}\cdot\boldsymbol{\varepsilon}_1+\cdots=0; \quad (25)$$

vibrating state,

$$(\omega_p^2-\omega^2)M(\mathbf{u}_1)\cdot\mathbf{u}_1+3\xi_s\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1)-\bar{\xi}\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1)+\xi_s^2[2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2v},\mathbf{u}_1)$$
$$+2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2s},\mathbf{u}_1)+\boldsymbol{\sigma}_{2s}\cdot L_2(\mathbf{u}_1)+\boldsymbol{\sigma}_{2v}\cdot L_2(\mathbf{u}_1)]-\bar{\xi}\xi_s\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_{2v},\mathbf{u}_1)$$
$$-\bar{\xi}^2\boldsymbol{\sigma}_{2v}\cdot L_2(\mathbf{u}_1)+\cdots=0. \quad (26)$$

These equations are used to derive simple expressions for the imperfection sensitivity of the vibration frequencies for nonsymmetric and symmetric structures. A perfect structure will here be denoted symmetric if the response of the structure is independent of the sign of the deformations.

2.2. *Nonsymmetric structures*

Equations (25) and (26) includes an infinite number of terms. To study the effects of imperfections on the vibration frequency only the terms with the lowest deflection order will be considered. We neglect the second and higher order deformations, i.e.

$$\mathbf{u}_{is}=0, \quad i=2,3,\ldots. \quad (27)$$

Using eqns (7) and (13), the following approximation then becomes valid

$$\boldsymbol{\sigma}_{2s}\cdot\boldsymbol{\varepsilon}_1=\tfrac{1}{2}\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1). \quad (28)$$

The vibration frequency of the imperfect structure is found in four steps. First the non-buckled state of equilibrium of the perfect structure $\bar{\mathbf{u}}$ is determined from (23). Then the vibration mode $\mathbf{u}_1$ and frequency ω_p are found from (24). The influence of small imperfections on the static mode amplitude parameter ξ_s is then determined from eqn (25), which will be rewritten as

$$\omega_p^2\xi_s-c\bar{\xi}+\tfrac{3}{2}a_1\xi_s^2-a_1\xi_s\bar{\xi}-\tfrac{1}{2}a_1\bar{\xi}^2=0 \quad (29)$$

where

$$c=\boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}_1/M(\mathbf{u}_1)\cdot\mathbf{u}_1$$
$$a_1=\boldsymbol{\sigma}_1\cdot L_2(\mathbf{u}_1)/M(\mathbf{u}_1)\cdot\mathbf{u}_1.$$

Finally, the vibration frequency of the imperfect structure is found from eqn (26), rewritten as

$$\omega^2=\omega_p^2+3a_1\xi_s-a_1\bar{\xi}. \quad (30)$$

Note that $\mathbf{u}_{2s}$ and $\mathbf{u}_{2v}$ are not needed in this lowest order approximation. Naturally, higher order approximations can also be applied. The investigation of vibrations of unloaded imperfect cylindrical panels by Hui (1984b) represents such a higher order analysis.

2.3. *Symmetric structures*

For symmetric structures such as straight beams, flat plates and axisymmetric shells in nonbreathing modes, the nonbuckled state of equilibrium of the perfect structure $\bar{\mathbf{u}}$ is also determined from (23). The vibration mode $\mathbf{u}_1$ and frequency ω_p are then found from (24). The influence of small geometrical imperfections on the vibration frequency cannot be estimated using eqns (29) and (30) since the coefficient a_1 is zero. In the next approximation, where second order effects of imperfections are taken into account, $\mathbf{u}_{2s}$ and $\mathbf{u}_{2v}$ are needed. We neglect the third and higher order deformations

$$\mathbf{u}_{is} = 0, \quad i = 3, 4, \ldots. \tag{31}$$

Using eqns (7) and (13), the following approximation then becomes valid

$$\boldsymbol{\sigma}_{3s} \cdot \boldsymbol{\varepsilon}_1 = \boldsymbol{\sigma}_1 \cdot L_{11}(\mathbf{u}_{2s}, \mathbf{u}_1). \tag{32}$$

Since a change of sign of the amplitudes should not have any consequence in (17) and (18) when $\hat{\mathbf{u}} = 0$, the symmetry results in the following equations for $\mathbf{u}_{2s}$ and $\mathbf{u}_{2v}$

$$\boldsymbol{\sigma}_{2s} \cdot \delta\bar{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}_1 \cdot L_{11}(\mathbf{u}_1, \delta\mathbf{u}) + \bar{\boldsymbol{\sigma}} \cdot L_{11}(\mathbf{u}_{2s}, \delta\mathbf{u}) = 0 \tag{33}$$

and

$$-\omega^2 M(\mathbf{u}_{2v}) \cdot \delta\mathbf{u} + \boldsymbol{\sigma}_{2v} \cdot \delta\bar{\boldsymbol{\varepsilon}} + 2\boldsymbol{\sigma}_1 \cdot L_{11}(\mathbf{u}_1, \delta\mathbf{u}) + \bar{\boldsymbol{\sigma}} \cdot L_{11}(\mathbf{u}_{2v}, \delta\mathbf{u}) = 0 \tag{34}$$

where $\delta\mathbf{u}$ is any kinematic admissible deflection field.

With eqns (31) and (32), eqns (25) and (26) reduce to

$$\omega_p^2 \xi_t - c\xi + (2b_1 + b_2)\xi_t^3 - b_1\hat{\xi}\xi_t^2 - b_2\xi_t\hat{\xi}^2 - b_1\hat{\xi}^3 = 0 \tag{35}$$

and

$$\omega^2 = \omega_p^2 + (2b_1 + b_2 + 2b_3 + b_4)\xi_t^2 - b_3\hat{\xi}\xi_t - b_2\hat{\xi}^2 \tag{36}$$

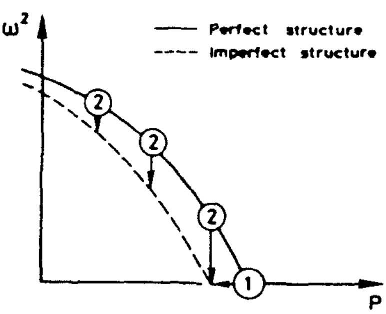
where

$$\begin{aligned}
c &= \boldsymbol{\sigma}_1 \cdot \boldsymbol{\varepsilon}_1 / M(\mathbf{u}_1) \cdot \mathbf{u}_1 \\
b_1 &= \boldsymbol{\sigma}_1 \cdot L_{11}(\mathbf{u}_{2s}, \mathbf{u}_1) / M(\mathbf{u}_1) \cdot \mathbf{u}_1 \\
b_2 &= \boldsymbol{\sigma}_{2s} \cdot L_2(\mathbf{u}_1) / M(\mathbf{u}_1) \cdot \mathbf{u}_1 \\
b_3 &= \boldsymbol{\sigma}_1 \cdot L_{11}(\mathbf{u}_{2v}, \mathbf{u}_1) / M(\mathbf{u}_1) \cdot \mathbf{u}_1 \\
b_4 &= \boldsymbol{\sigma}_{2v} \cdot L_2(\mathbf{u}_1) / M(\mathbf{u}_1) \cdot \mathbf{u}_1.
\end{aligned}$$

3. DISCUSSION

In the present theory only one vibration mode is assumed to exist together with an imperfection in this mode. Many structures have multiple vibration modes, and for these structures similar equations can be derived by considering the imperfections to be linear combinations of these modes, as shown for buckling of axially stiffened cylindrical shells by Byskov and Hutchinson (1977).

Fitch (1968) has derived simple equations for the influence of imperfections on the buckling load of slender structures. The present equations are concerned with the influence



1 Reduction according to the theory by Fitch (1968).
2 Reduction according to the present theory.

Fig. 1. Evaluation of the squared frequency and the buckling load of an imperfect structure.

of imperfections on the frequency rather than on the load. Since the present equations are valid during the loading of the imperfect structure, in principle buckling can also be investigated with the present equations (see Fig. 1).

It is emphasized that when evaluating the vibration frequency of an imperfect structure with the present equations the first and second order modes $\mathbf{u}_1$ and $\mathbf{u}_{2r}$ are approximated with the modes of the perfect structure including the inertia terms corresponding to the vibration frequency of the perfect structure.

Elishakoff *et al.* (1987) studies the influence of initial geometric imperfections on the vibration frequencies of cylindrical panels. By using the postbuckling coefficient, see Budiansky (1966) and Fitch (1968), together with analytical results for a single degree-of-freedom structure, they derive simple equations for the imperfection sensitivity of different panels. The present theory differs from the approach of Elishakoff *et al.* (1987) by including inertia terms in the second order analysis and by including the imperfection in a way so that also the vibration frequency of the unloaded structure can be investigated. For symmetric vibrations of unloaded cylindrical panels, the present analysis procedure is similar to the procedure used by Hui (1984b).

4. EXAMPLES

The theory will now be illustrated by application to three different problems.

First, the vibrations of a beam, which is simply supported and loaded by a forced axial displacement of one of the supports, are investigated. The governing equations are solved analytically and the sensitivity of the fundamental vibration frequency to imperfections is shown graphically.

Then the vibrations of a simply supported plate are investigated. The plate is loaded in one direction and the edges are assumed to remain straight during loading and vibration. The governing equations are solved analytically and the imperfection sensitivity of the fundamental vibration frequency is shown graphically for a square plate.

Finally, the vibrations of a truncated conical shell are investigated. The edges are clamped and assumed to move axially during loading. The governing equations are solved numerically using a finite difference computer program, and the sensitivity of the vibration frequency to imperfections is derived.

4.1. *Application to a beam*

A simply supported straight beam of length d and uniform cross-section is loaded by a forced axial displacement v_d of one of the supports (see Fig. 2). We will use the present approach to study the influence of an initial geometrical imperfection with shape of the vibration mode on the fundamental vibration frequency.

Let v and w denote the deflections in the axial and transverse directions, respectively

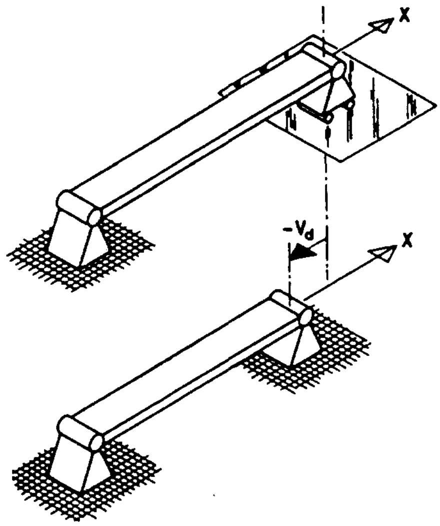


Fig. 2. A simply supported beam loaded by a forced displacement of one of the supports.

$$\mathbf{u} = \{v; w\}. \tag{37}$$

The generalized strains $\boldsymbol{\varepsilon}$ are the strain ε and the curvature κ approximated by

$$\boldsymbol{\varepsilon} = \{\varepsilon; \kappa\} = L_1(\mathbf{u}) + \tfrac{1}{2}L_2(\mathbf{u}) \tag{38}$$

where

$$L_1(\mathbf{u}) = \{v_{,x}; w_{,xx}\}$$
$$L_2(\mathbf{u}) = \{(w_{,x})^2; 0\}. \tag{39}$$

With eqn (5) we find

$$L_{11}(\mathbf{u}_a, \mathbf{u}_b) = \{w_{a,x}w_{b,x}; 0\}. \tag{40}$$

The generalized stresses are the tension n and the bending moment m. These are found from the elastic relation

$$\boldsymbol{\sigma} = \{n; m\} = H(\boldsymbol{\varepsilon}) = \{EA\varepsilon; EI\kappa\} \tag{41}$$

where E, A, and I are Young's modulus, cross-sectional area, and moment of inertia, respectively.

Ignoring axial inertia the mass operator becomes

$$M(\mathbf{u}) = \rho A\{0; w\}, \tag{42}$$

where ρ is the mass density of the beam material.

The boundary conditions can be written as

$$v(0,t) = w(0,t) = w(d,t) = w_{,xx}(0,t) = w_{,xx}(d,t) = 0, \quad v(d,t) = v_d. \tag{43}$$

We will expand v, w, n and m in the perturbational form employed in eqns (9) and (12).

Differential equations for the nonbuckled static state $\bar{\mathbf{u}}$ for the perfect structure are obtained by inserting expressions (37)–(41) and eqn (16), in (23)

$$\int_0^d [EA(\bar{v}_{,x}+\tfrac{1}{2}(\bar{w}_{,x})^2)(\delta v_{,x}+\bar{w}_{,x}\delta w_{,x})+EI\bar{w}_{,xx}\delta w_{,xx}]\,dx = 0, \tag{44}$$

and, using the Euler–Lagrange equations, the following equations are obtained

$$(\bar{v}_{,x}+\tfrac{1}{2}(\bar{w}_{,x})^2)_{,x} = 0; \quad -EA(\bar{v}_{,x}\bar{w}_{,x}+\tfrac{1}{2}(\bar{w}_{,x})^3)_{,x}+EI\bar{w}_{,xxxx} = 0. \tag{45}$$

With the present boundary conditions we find

$$\bar{v}_{,x}(x) = \frac{v_d}{d}, \quad \bar{w}(x) = 0. \tag{46}$$

We obtain the following differential equations for the vibration mode $\mathbf{u}_1$ and the frequency ω_p of the perfect beam, again using the Euler–Lagrange equations together with eqn (24)

$$EIw_{1,xxxx}-\frac{EAv_d}{d}w_{1,xx}-\omega_p^2\rho Aw_1 = 0, \quad v_{1,xx} = 0. \tag{47}$$

With the present boundary conditions the solution becomes

$$w_1(x) = r_g \sin\left(\frac{\pi x}{d}\right), \quad v_1(x) = 0, \quad m_1(x) = -r_g EI\left(\frac{\pi}{d}\right)^2 \sin\left(\frac{\pi x}{d}\right)$$

$$\omega_p^2 = \frac{EI}{\rho A}\left(\frac{\pi}{d}\right)^4+\frac{Ev_d}{\rho d}\left(\frac{\pi}{d}\right)^2. \tag{48}$$

The radius of gyration $r_g = \sqrt{I/A}$ is arbitrarily chosen to normalize the amplitude.

With (46) and (48) inserted in (33) and (34) we obtain differential equations for the second order modes $\mathbf{u}_{2s}$ and $\mathbf{u}_{2r}$ using the Euler–Lagrange equations,

$$[v_{2s,x}+\tfrac{1}{2}(w_{1,x})^2]_{,x} = 0, \quad w_{2s} = 0$$

$$[v_{2r,x}+(w_{1,x})^2]_{,x} = 0, \quad w_{2r} = 0. \tag{49}$$

The present boundary conditions have the property that

$$\int_0^d v_{2s,x}\,dx = \int_0^d v_{2r,x}\,dx = 0. \tag{50}$$

Now n_{2s} and n_{2r} can be found from (13), (14), (39)–(41), (49) and (50):

$$n_{2s} = EA[v_{2s,x}+\tfrac{1}{2}(w_{1,x})^2] = \frac{EA}{2d}\int_0^d (w_{1,x})^2\,dx = \frac{EI}{4}\left(\frac{\pi}{d}\right)^2$$

$$n_{2r} = EA[v_{2r,x}+(w_{1,x})^2] = \frac{EA}{d}\int_0^d (w_{1,x})^2\,dx = \frac{EI}{2}\left(\frac{\pi}{d}\right)^2. \tag{51}$$

It is emphasized that the stresses n_{2s} and n_{2r} are caused by the axially fixed boundary condition.

With the generalized stresses m_1, n_{2s}, n_{2r} and the mode shape w_1 known the coefficients in (35) and (36) can be calculated. We get

$$c = \frac{\int_0^d m_1 w_{1,xx}\,\mathrm{d}x}{\int_0^d \rho A (w_1)^2\,\mathrm{d}x} = \frac{\int_0^d EI\left(\frac{\pi}{d}\right)^4 \sin^2\left(\frac{\pi x}{d}\right)\mathrm{d}x}{\int_0^d \rho A \sin^2\left(\frac{\pi x}{d}\right)\mathrm{d}x} = \frac{EI}{\rho A}\left(\frac{\pi}{d}\right)^4$$

$$b_1 = 0$$

$$b_2 = \frac{\int_0^d n_{2s}(w_{1,x})^2\,\mathrm{d}x}{\int_0^d \rho A (w_1)^2\,\mathrm{d}x} = \frac{\int_0^d \frac{EI}{4}\left(\frac{\pi}{d}\right)^4 \sin^2\left(\frac{\pi x}{d}\right)\mathrm{d}x}{\int_0^d \rho A \sin^2\left(\frac{\pi x}{d}\right)\mathrm{d}x} = \frac{EI}{4\rho A}\left(\frac{\pi}{d}\right)^4$$

$$b_3 = 0$$

$$b_4 = \frac{\int_0^d n_{2t}(w_{1,x})^2\,\mathrm{d}x}{\int_0^d \rho A (w_1)^2\,\mathrm{d}x} = \frac{\int_0^d \frac{EI}{2}\left(\frac{\pi}{d}\right)^4 \sin^2\left(\frac{\pi x}{d}\right)\mathrm{d}x}{\int_0^d \rho A \sin^2\left(\frac{\pi x}{d}\right)\mathrm{d}x} = \frac{EI}{2\rho A}\left(\frac{\pi}{d}\right)^4. \tag{52}$$

Then the governing equations for the static mode amplitude normalized with the radius of gyration ξ_t and the normalized vibration frequency ω_r can be written as

$$\xi_t(1-P_r) - \bar{\xi} + \tfrac{1}{4}\xi_t^3 - \tfrac{1}{4}\xi_t\bar{\xi}^2 = 0 \tag{53}$$

and

$$\omega_r^2 = 1 - P_r + \tfrac{3}{4}\xi_t^2 - \tfrac{1}{4}\bar{\xi}^2. \tag{54}$$

In these equations, the applied load from the forced displacement v_d is normalized with the critical load of the perfect structure

$$P_r = -\frac{Ev_d}{\rho d}\left(\frac{d}{\pi}\right)^2, \tag{55}$$

and the frequency ω is normalized with the frequency of the unloaded perfect structure

$$\omega_r^2 = \omega^2\frac{\rho A}{EI}\left(\frac{d}{\pi}\right)^4. \tag{56}$$

These results are in full agreement with the results obtained by a two-term Galerkin approximation described by Elishakoff *et al.* (1985).

Figure 3 shows graphs of the nondimensional vibration frequency ω_r versus the load ratio P_r for different values of the imperfection amplitude normalized with the radius of gyration $\bar{\xi}$. Contour lines for constant amplitude of the normalized static deflection $\xi_s = \xi_t - \bar{\xi}$ are shown too. The vibration frequency is significantly raised by the imperfections, when the load approaches the critical value. The imperfections also raise the vibration frequency for zero load. When the beam is in tension the effect of imperfections becomes negligible for increasing tensile load.

4.2. *Application to a rectangular plate*

A simply supported rectangular plate of sidelengths a and b and thickness h is preloaded in one direction (see Fig. 4). The supports are assumed free to move in-plane, and the edges to remain straight.

Let $\mathbf{u}$ denote the displacement vector with components in the x, y and z direction

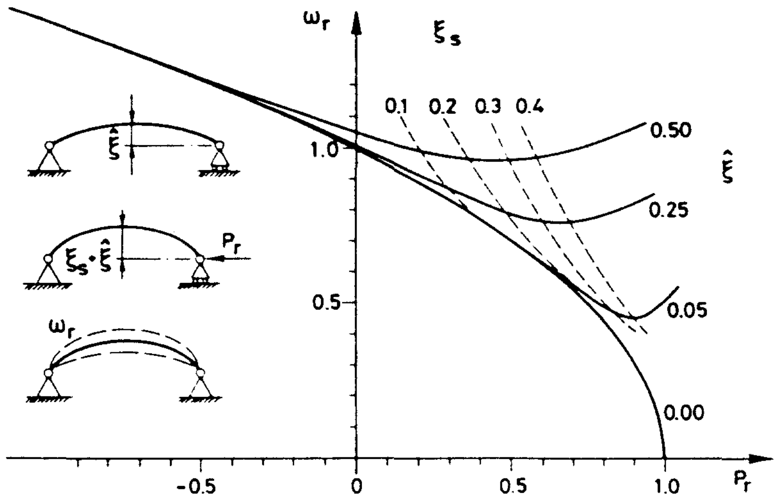


Fig. 3. Vibration frequency of an imperfect beam.

$$\mathbf{u} = \{u, v, w\}. \tag{57}$$

The generalized strains ε are of the form

$$\varepsilon = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}, \kappa_{xx}, \kappa_{yy}, \kappa_{xy}\} = L_1(\mathbf{u}) + \tfrac{1}{2}L_2(\mathbf{u}), \tag{58}$$

where the linear part of the strains is approximated by

$$L_1(\mathbf{u}) = \{u_{,x}\,;\, v_{,y}\,;\, \tfrac{1}{2}(u_{,y}+v_{,x})\,;\, w_{,xx}\,;\, w_{,yy}\,;\, w_{,xy}\}, \tag{59}$$

and the bilinear part of the strains is approximated by

$$L_2(\mathbf{u}) = \{(w_{,x})^2\,;\,(w_{,y})^2\,;\, w_{,x}w_{,y}\,;\,0\,;\,0\,;\,0\}. \tag{60}$$

With eqn (5) we find that

$$L_{11}(\mathbf{u}_a, \mathbf{u}_b) = \{w_{a,x}w_{b,x}\,;\, w_{a,y}w_{b,y}\,;\, \tfrac{1}{2}(w_{a,x}w_{b,y} + w_{a,y}w_{b,x})\,;\,0\,;\,0\,;\,0\}. \tag{61}$$

Hooke's law is used to obtain the generalized stresses

$$\begin{aligned} \sigma &= \{n_{xx}, n_{yy}, n_{xy}, m_{xx}, m_{yy}, m_{xy}\} = H(\varepsilon) \\ &= \frac{D}{h^2}\{12(\varepsilon_{xx}+v\varepsilon_{yy})\,;\,12(\varepsilon_{yy}+v\varepsilon_{xx})\,;\,12(1-v)\varepsilon_{xy}\,; \\ &\quad h^2(\kappa_{xx}+v\kappa_{yy})\,;\,h^2(\kappa_{yy}+v\kappa_{xx})\,;\,h^2(1-v)\kappa_{xy}\}, \end{aligned} \tag{62}$$

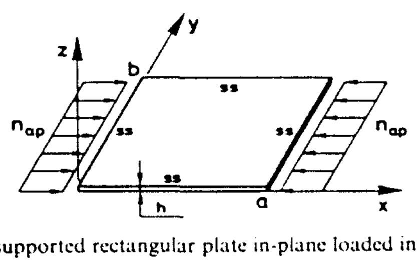


Fig. 4. A simply supported rectangular plate in-plane loaded in one direction.

where ν is Poisson's ratio, and D is the flexural stiffness of the plate

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{63}$$

Ignoring in-plane inertia, the mass operator M is

$$M(\mathbf{u}) = \rho h\{0;0;w\}. \tag{64}$$

The boundary conditions for the present problem can be written as

$$w(x,0) = w(x,b) = w(0,y) = w(a,y) = 0$$

$$\int_0^b n_{xx}\,dy = -n_{ap}b, \quad \int_0^a n_{yy}\,dx = \int_0^b n_{xy}\,dy = 0$$

$$v_{,x}(x,0) = v_{,x}(x,b) = u_{,y}(0,y) = u_{,y}(a,y) = 0 \tag{65}$$

where n_{ap} is the external applied load to the edges.

Let Φ be the Airy stress function for the membrane stresses. When inserting eqns (57)–(64) in the eqn of equilibrium (1) the von Karman plate equations can be obtained using the divergence theorem

$$\rho h w_{,tt} + D\nabla^4 w = \Phi_{,yy}w_{,xx} + \Phi_{,xx}w_{,yy} - 2\Phi_{,xy}w_{,xy} \tag{66}$$

and

$$\frac{1}{Eh}\nabla^4\Phi = w_{,xy}^2 - w_{,xx}w_{,yy}. \tag{67}$$

The Airy stress function Φ and the operator $\nabla^4(\ )$ are defined by

$$\Phi_{,yy} = n_{xx};\quad \Phi_{,xx} = n_{yy};\quad \Phi_{,xy} = -n_{xy};\quad \nabla^4(\ ) = \nabla^2(\nabla^2(\ ));\nabla^2(\ ) = (\ )_{,xx}+(\ )_{,yy}. \tag{68}$$

We will expand the deformations, the strains, and the stresses in accordance with eqns (9) and (12).

The nonbuckled response $\bar{\mathbf{u}}$ of the perfect plate subjected to the applied load is found by inserting eqns (57)–(63) in (23) and using the divergence theorem together with the boundary conditions. We get the well known result

$$\bar{u}_{,x} = -\frac{n_{ap}}{Eh}, \quad \bar{v}_{,y} = \nu\frac{n_{ap}}{Eh}, \quad \bar{w} = 0. \tag{69}$$

The following equation is obtained for the mode $\mathbf{u}_1$ either by inserting (57)–(64) in (24) and using the divergence theorem, or by setting $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}_1 \cos\omega_p t$ in the von Karman equations:

$$-\omega_p^2\rho h w_1 + D\nabla^4 w_1 = n_{ap}w_{1,xx};\quad \nabla^4\Phi_1 = 0. \tag{70}$$

This gives with the present boundary conditions

$$\mathbf{u}_1 = \left\{0;0;h\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right\} \tag{71}$$

and

$$\omega_p^2 = \frac{D\pi^4}{\rho h}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 - n_{xp}\frac{\pi^2}{\rho h}\frac{1}{a^2} \tag{72}$$

where the plate thickness h is arbitrarily chosen for normalizing the deflections.

The equations governing the static Φ_{2s} and vibrating Φ_{2v} second order Airy stress functions can be obtained from (33) and (34) by use of the divergence theorem

$$\frac{1}{Eh}\nabla^4\Phi_{2s} = w_{1,xy}^2 - w_{1,xx}w_{1,yy} = \frac{1}{2}\left(\frac{\pi^2}{ab}\right)^2 h^2\left(\cos\frac{2\pi x}{a}+\cos\frac{2\pi y}{b}\right)$$

$$\frac{1}{Eh}\nabla^4\Phi_{2v} = 2(w_{1,xy}^2 - w_{1,xx}w_{1,yy}) = \left(\frac{\pi^2}{ab}\right)^2 h^2\left(\cos\frac{2\pi x}{a}+\cos\frac{2\pi y}{b}\right). \tag{73}$$

With the actual boundary conditions the solutions are

$$\Phi_{2s} = \frac{Eh^3}{32a^2b^2}\left(a^4\cos\frac{2\pi x}{a}+b^4\cos\frac{2\pi y}{b}\right)$$

$$\Phi_{2v} = \frac{Eh^3}{16a^2b^2}\left(a^4\cos\frac{2\pi x}{a}+b^4\cos\frac{2\pi y}{b}\right). \tag{74}$$

Now the coefficients in (35) and (36) can be calculated:

$$c = \frac{\int_0^a\int_0^b [m_{1xx}w_{1,xx} + m_{1yy}w_{1,yy} + 2m_{1xy}w_{1,xy}]\,\mathrm{d}x\,\mathrm{d}y}{\int_0^a\int_0^b \rho h(w_1)^2\,\mathrm{d}x\,\mathrm{d}y}$$

$$= \frac{Dh^2\pi^4\int_0^a\int_0^b\left[\left(\frac{1}{a^4}+\frac{2\nu}{a^2b^2}+\frac{1}{b^4}\right)\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}+\frac{2-2\nu}{a^2b^2}\cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}\right]\mathrm{d}x\,\mathrm{d}y}{\int_0^a\int_0^b \rho h^3\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y}$$

$$= \frac{D\pi^4}{\rho h}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2$$

$$b_1 = 0$$

$$b_2 = \frac{\int_0^a\int_0^b [\Phi_{2s,yy}(w_{1,x})^2 + \Phi_{2s,xx}(w_{1,y})^2]\,\mathrm{d}x\,\mathrm{d}y}{\int_0^a\int_0^b \rho h(w_1)^2\,\mathrm{d}x\,\mathrm{d}y}$$

$$= \frac{\frac{Eh^5\pi^4}{8}\int_0^a\int_0^b\left[\frac{-1}{a^4}\cos\frac{2\pi y}{b}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}-\frac{1}{b^4}\cos\frac{2\pi x}{a}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}\right]\mathrm{d}x\,\mathrm{d}y}{\int_0^a\int_0^b \rho h^3\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y}$$

$$= \tfrac{3}{4}(1-\nu^2)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)\frac{D\pi^4}{\rho h}$$

$$b_3 = 0$$

$$b_4 = \frac{\displaystyle\int_0^a \int_0^b [\Phi_{2r,yy}(w_{1,x})^2 + \Phi_{2r,xx}(w_{1,y})^2]\, dx\, dy}{\displaystyle\int_0^a \int_0^b \rho h (w_1)^2\, dx\, dy}$$

$$= \frac{\displaystyle\frac{Eh^5\pi^4}{4}\int_0^a \int_0^b \left[\frac{-1}{a^4}\cos\frac{2\pi y}{b}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b} - \frac{1}{b^4}\cos\frac{2\pi x}{a}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}\right] dx\, dy}{\displaystyle\int_0^a \int_0^b \rho h^3 \sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\, dx\, dy}$$

$$= \tfrac{3}{2}(1-\nu^2)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)\frac{D\pi^4}{\rho h}. \tag{75}$$

The governing equations for static deflection (35) and vibration frequency (36) then become

$$\frac{1}{4}\left(1+\frac{a^2}{b^2}\right)^2(\xi_t - \hat{\xi}) - P_r\xi_t + \frac{3}{16}(1-\nu^2)\left(1+\frac{a^4}{b^4}\right)(\xi_t^3 - \hat{\xi}^2) = 0 \tag{76}$$

and

$$\omega_r^2 = \frac{1}{4}\left(1+\frac{a^2}{b^2}\right)^2 - P_r + \frac{3}{16}(1-\nu^2)\left(1+\frac{a^4}{b^4}\right)(3\xi_t^2 - \hat{\xi}^2), \tag{77}$$

where

$$P_r = \frac{a^2}{4D\pi^2} n_{ap}$$

$$\omega_r^2 = \frac{\rho h}{4D}\left(\frac{a}{\pi}\right)^4 \omega^2. \tag{78}$$

The result is identical to the result found by Hui and Leissa (1983).

Figure 5 shows the nondimensional vibration frequency ω_r of a square plate as a function of the uniaxially load ratio P_r for different values of the imperfection amplitude

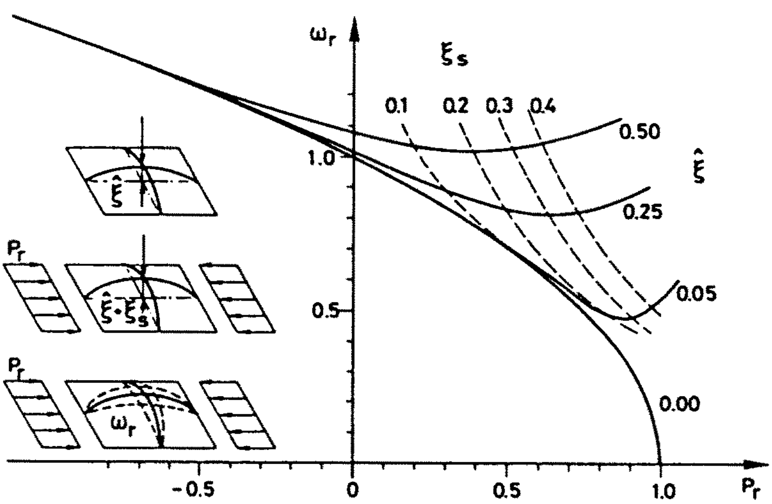


Fig. 5. Vibration frequency of an imperfect square plate.

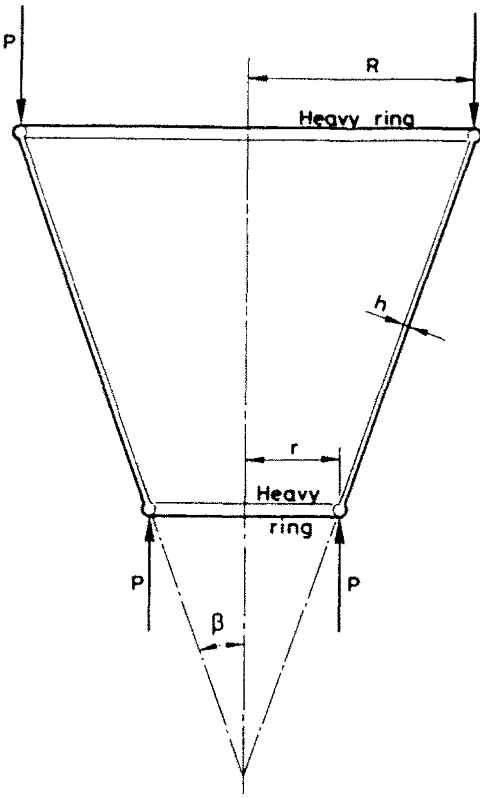


Fig. 6. The clamped truncated conical shell.

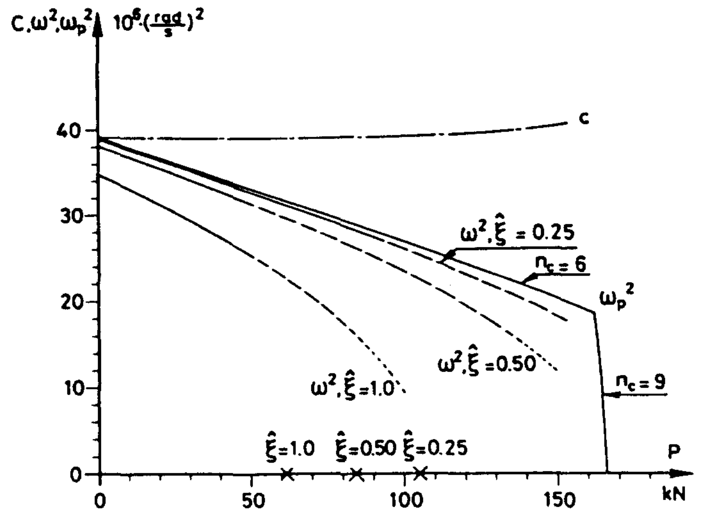
normalized with the plate thickness $\bar{\xi}$. Contour lines show the corresponding amplitudes of the normalized static deflection $\xi_s = \xi_t - \bar{\xi}$. The effect of imperfections shows the same trends as was seen for a beam.

It is noted that the neglect of third and higher order terms, eqn (31), is not valid for large buckling amplitudes. Ilanko and Dickinson (1987a) have applied a Rayleigh Ritz procedure to the present problem. They show that the effect of higher order terms becomes important when the amplitude of the buckling mode become larger than about three times the plate thickness. Their theoretical results are validated experimentally, see Ilanko and Dickinson (1987b).

4.3. *Application to a truncated conical shell*

A truncated conical shell, clamped in both ends, is considered (see Fig. 6). The ends can move in the axial direction during static loading. The present theory will be used to study the influence of an initial geometrical imperfection with shape of a vibration mode on the vibration frequency associated with this mode. For this purpose a finite difference computer program for general analysis of axisymmetric shells with a nonlinear prebuckling state has been adjusted to cover the present theory. This computer program has been described and applied to buckling of LNG spheres by Pedersen and Jensen (1976). In the special case of axisymmetric loading, the numerical procedures in the computer program are, apart from the inertia terms, identical to the procedure in the computerized buckling and postbuckling analysis of spherical caps described by Fitch (1968). In the same reference, detailed information on the interpretation of the tensor terms in the governing equations and coefficients can be found in case of axisymmetric shells.

The geometry and material properties of the shell are the same as considered in Tani (1974). The semi-vertex angle $\beta = 20°$, the smaller end radius $r = 54.1$ mm, the larger end radius $R = 124$ mm, and the shell thickness $h = 0.050$ mm. The mass density of the shell



—— Squared frequency for imperfect shell according to the present theory.
× Buckling load for imperfect shell according to the theory by Fitch (1968).

Fig. 7. Squared vibration frequencies of perfect and imperfect shells and the coefficient c.

material $\rho = 8310$ kg m^{-3}, Young's modulus $E = 206$ GN m^{-2}, and the Poisson ratio $\nu = 0.30$.

The results from the calculations are shown in Figs 7 and 8. The square of the frequency of the perfect structure ω_p^2 is shown in Fig. 7. The mode with a circumferential wave number $n_c = 6$ has the lowest vibration frequency at applied axial loads up to $P = 163$ kN. At this load the lowest frequency mode changes to $n_c = 9$, which also becomes the buckling mode at the load $P = 166$ kN. The vibration mode for $P = 0$, and the buckling mode are shown in Fig. 9. The amplitude of $\mathbf{u}_1$ is normalized with the shell thickness and the coefficients c, b_1, b_2, b_3 and b_4 in (35) and (36) are calculated. The coefficient c is shown in Fig. 7. It is noted that c represents the square vibration frequency in mode $\mathbf{u}_1$ for the unloaded shell. The variation of c with the applied axial load is caused by the variation of the mode shape $\mathbf{u}_1$ with the load. The coefficients b_1, b_2, b_3 and b_4 are shown in Figure 8. All these coefficients vary smoothly with the load up to $P = 155$ kN. Above this load the coefficients and the lowest frequency mode shape changes rapidly with the load, and in this region the present

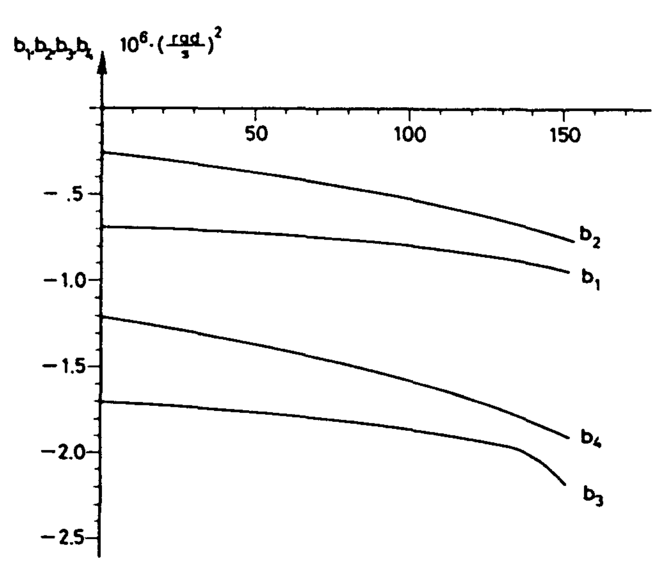
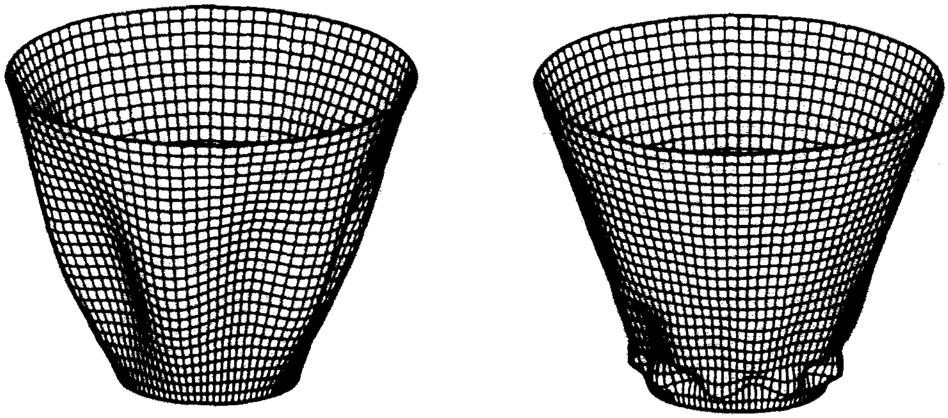


Fig. 8. The coefficients b_1, b_2, b_3 and b_4 as functions of the applied load P.



Vibration mode Buckling mode

Fig. 9. Vibration mode for the unloaded shell and the buckling mode.

theory is not suitable for estimating the influence of the imperfection on the vibration frequency.

The coefficients c, b_1, b_2, b_3 and b_4 together with the vibration frequency of the perfect structure ω_p are inserted in the eqns (33) and (34). From these equations are calculated frequencies shown in Fig. 7 for different amplitudes of $\bar{\xi}$, the imperfection amplitude normalized with the shell thickness h. Punctuation of the curves indicates that the theory is restricted to moderate reductions in the frequency. We see that the imperfections lower the vibration frequency significantly, and that this effect becomes more pronounced for higher loads.

Fitch (1968) uses the following equation to estimate the influence of an initial geometric imperfection with shape of the buckling mode on the buckling load of a symmetric structure with negative value of the postbuckling coefficient b:

$$\left(1-\frac{\lambda_s}{\lambda_c}\right)^{3/2} = \frac{3\sqrt{3}}{2}|\bar{\xi}|\alpha\sqrt{-b}\left(\frac{\lambda_s}{\lambda_c}\right). \tag{79}$$

In the equation, α, λ_s, and λ_c denote the nonlinearity coefficient of the prebuckling state, and the critical load of the imperfect and perfect structure, respectively. The equation is asymptotically correct for small values of $\bar{\xi}$, the amplitude of the imperfection normalized with the shell thickness.

For the present shell, the coefficients α and b are calculated numerically using the method described in Fitch (1968) and in Pedersen and Jensen (1976). We find

$$\alpha = 0.57$$
$$b = -0.83. \tag{80}$$

These values are inserted in eqn (79) to obtain the results also shown in Fig. 7. We see that also the buckling load can be seriously lowered by imperfections. A direct comparison of the two theories is not possible, since they apply different mode shapes for the present shell.

It is noted that both the present theory and eqn (79) imply that the vibration frequency or the buckling load corresponds to a unique deflection mode shape. For the present case, several eigenfrequencies can be found in the closeness of the lowest. Furthermore, the nonlinear static response of the perfect structure shows limit load behavior at a load which is a few per cent higher than the lowest bifurcation load. The validity of the results are therefore questionable, but they are at least believed to be representative for the vibration behaviour of imperfect shells.

5. CONCLUSION

A theory for the analysis of the effect of initial geometric imperfections on the vibration behaviour of undamped, conservatively loaded, linear elastic beam and shell structures is presented. The theory is restricted to structures where the imperfections are of the same shape as the vibration mode. Simple equations are derived for nonsymmetric and symmetric structures.

The theory is illustrated by application to analysis of vibrations of a beam, a rectangular plate, and a truncated conical shell. The results show that the vibration frequency of a beam or a plate may be significantly raised by geometrical imperfections. This has previously been found also by other investigators. On the other hand, the numerical results show that the vibration frequency for a truncated conical shell may be significantly lowered due to geometrical imperfections.

Even though the theory is restricted to geometrical imperfections with the same shape as the vibration mode, the results will be representative for other shapes and types of imperfections.

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